STATIONARY MOTION OF A SYSTEM OF PARALLEL LONGITUDINAL SHEAR CRACKS

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The stationary motion of a rectilinear semiinfinite crack in an infinite elastic body was considered in [1, 2]. It will often be that not one, but several, cracks are propagated in a medium. In this connection, it is interesting to consider the motion of a system of semiinfinite parallel cracks. In this paper we limit ourselves to the consideration of the case of longitudinal shear cracks. A similar problem, in mathematical respects, about the steady motion of a rupture crack in a strip was studied in [3, 4].

Let us consider the motion of a system of semiinfinite parallel cracks, longitudinal shear slits with constant velocity. In an xOy coordinate system moving together with the cracks, the slit surfaces are a system of parallel half-lines $x, y \in S$, where

$$S = \{x < 0,\ y = d(2n+1),\ n = 0,\ \pm 1,\ \pm 2, \ldots\}.$$

Let the rate of crack growth V be less than the transverse wave velocity c in the medium. It is also assumed that the crack motion is stationary, i.e., the strain and stress are independent of the time in the moving coordinate system. In this case, the elasticity theory equations describing the problem have the form

$$\beta^2 \partial^2 w/\partial x^2 + \partial^2 w/\partial y^2 = 0,\ \tau = \mu \partial w/\partial y, \tag{1}$$

where $\beta^2 = 1 - V^2/c^2$; w is the displacement along the z axis, $\tau = \sigma_{yz}$ is the stress tensor component, and μ is the shear modulus. Let the very same homogeneous load

$$\tau(x, y) = -\tau_0,\ x, y \in S$$

be applied to the edges of all the cracks. Then the strains and stresses are periodic functions of the coordinate y with period 2d, and the problem is reduced to the construction of a solution of (1) in the domain $-d < y < d$ that satisfies the boundary conditions

$$\tau(x, \pm d) = -\tau_0,\ x < 0; \tag{2}$$

$$w(x, \pm d) = 0,\ x > 0. \tag{3}$$

To solve the problem formulated, we use the method proposed in [5-7]. We temporarily replace the inhomogeneous boundary condition (2) by a condition of the form

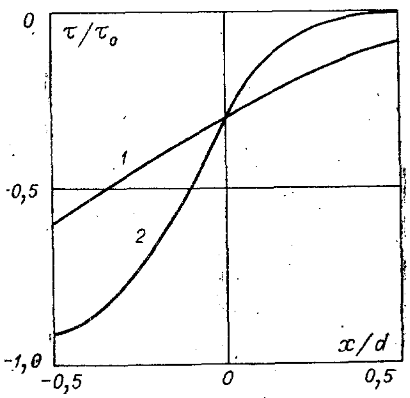

Fig. 1

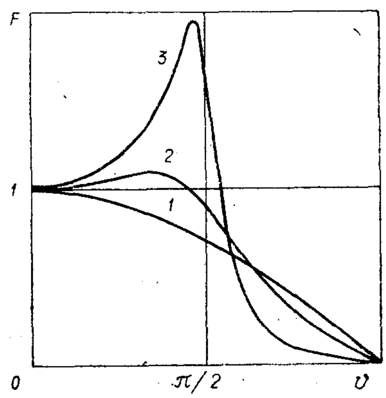

Fig. 2

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$$\tau(x, \pm d) = -\tau_0 e^{\alpha x}, \ x < 0, \tag{4}$$

where α is a positive number which should be set equal to zero in the final formulas. We represent the solution of (1) in the form

$$w(x, y) = -\frac{1}{2\pi\mu\beta} \int_{-\infty}^{\infty} \frac{1}{q} \frac{\operatorname{sh}(\beta q y)}{\operatorname{ch}(\beta q d)} C(q) e^{-iqx} dq, \tag{5}$$

$$\tau(x, y) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\operatorname{ch}(\beta q y)}{\operatorname{ch}(\beta q d)} C(q) e^{-iqx} dq,$$

where $C(q)$ is an unknown function. Taking into account that

$$e^{\alpha x} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-iqx}}{q - i\alpha} dq, \quad x < 0,$$

we obtain dual integral equations for $C(q)$

$$\int_{-\infty}^{\infty} K(\beta q d) C(q) e^{-iqx} dq = 0, \quad x > 0,$$

$$\int_{-\infty}^{\infty} \left\{ C(q) - \frac{1}{i} \frac{\tau_0}{q - i\alpha} \right\} e^{-iqx} dq = 0, \quad x < 0 \tag{6}$$

from boundary conditions (3), (4). Here, $K(z) = \pi \operatorname{th}(z)/z$. Representing the function $K(z)$ in the form $K(z) = K_+(z)K_-(z)$, where $K_+(z) = \Gamma(1/2 - iz/\pi)/\Gamma(1 - iz/\pi)$, $K_-(z) = K_+(-z)$, $\Gamma(z)$ is the Gamma function, and taking account of the analytic properties of the functions $K_+(z)$ and $K_-(z)$, it can easily be shown [5-7] that the solution of (6) has the form

$$C(q) = \frac{1}{i} \frac{\tau_0 K_+(i\beta\alpha d)}{(q - i\alpha) K_+(\beta q d)}.$$

Substituting the value found for $C(q)$ into (5), evaluating the integrals, and then letting the parameter α tend to zero, we obtain the solution of the initial problem. Omitting the intermediate computations, we present the final result:

$$\frac{w(x, y)}{w_0} = \frac{2}{\pi} \operatorname{Im}\left\{-z + \ln\left(e^z + \sqrt{1 + e^{2z}}\right)\right\}, \tag{7}$$

$$\frac{\tau(x, y)}{\tau_0} = \operatorname{Re}\left\{-1 + \frac{e^z}{\sqrt{1 + e^{2z}}}\right\},$$

where $w_0 = d\tau_0/\mu$, $z = \pi(x/\beta + iy)/2d$. It can be seen by direct substitution that the solution in the form (7) satisfies (1) and the boundary conditions (2) and (3). The distribution

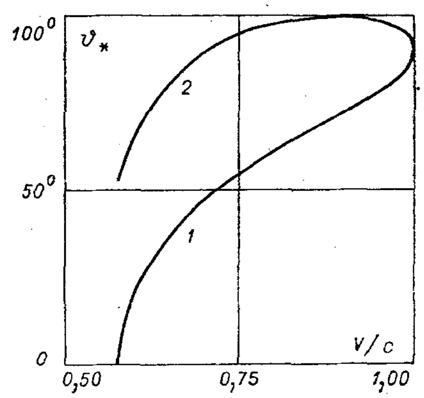

Fig. 3

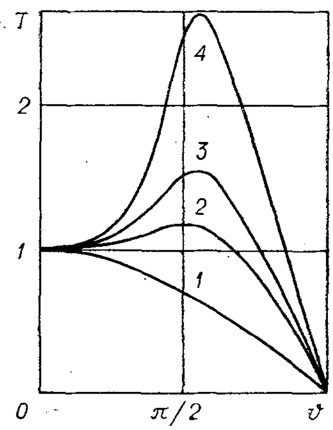

Fig. 4

of the stresses τ/τ_0 on the $y = 0$ axis is represented in Fig. 1. Curves 1 and 2 correspond to the values $V = 0$ and $0.95c$ for the crack front velocity.

Let us investigate the behavior of the solution near the apex of an individual crack. Let us introduce a r, θ polar coordinate system with origin at the crack apex, and the auxiliary variables ρ, φ, connected to the angle θ by the relation

$$\rho e^{i\varphi} = \cos(\theta) + i\beta \sin(\theta), \quad -\pi < \theta < \pi,$$

where the values $\theta = \pm\pi$ correspond to the crack edges.

It follows from (7) that near the crack apex the following asymptotic relations are valid [8]

$$w(r, \theta) = \frac{K}{\mu}\sqrt{2r}\,\frac{\sqrt{\rho}}{\beta}\sin\left(\frac{\varphi}{2}\right), \quad \tau(r, \theta) = \frac{K}{\sqrt{2r}}F(\theta),$$

where $F(\theta) = \cos(\varphi/2)/\sqrt{\rho}$, while the expression

$$K = \tau_0\sqrt{\frac{2d}{\pi}}\left(1 - \frac{V^2}{c^2}\right)^{1/4} \tag{8}$$

is obtained for the stress intensity factor. Graphs of the function $F(\theta)$ are presented in Fig. 2. Curves 1-3 correspond to the value $V = 0$, $0.8c$, and $0.985c$. For $V < c/\sqrt{3}$ the stress τ as a function of θ has a maximum directly on the continuation of the crack for $\theta = 0$. For $V > c/\sqrt{3}$, τ has two symmetric maximums for $\theta = \pm\theta_*$, where θ_* is the solution of the equation

$$\frac{V^2}{c^2} = \frac{\sqrt{1 + 8\cos^2\theta_*} - 1 - 2\cos^2\theta_*}{2\sin^2\theta_*\cos^2\theta_*}.$$

The dependence of θ_* on the parameter V/c is presented in Fig. 3 (curve 1). As V/c changes between $1/\sqrt{3}$ and 1 the value of θ_* varies between zero and 90°.

Analogously, for the stress tensor components $\sigma_{\theta z}$ near the crack apex, we have the asymptotic expression [9]

$$\sigma_{\theta z} = \frac{K}{\sqrt{2r}}T(\theta),$$

where

$$T(\theta) = \frac{1}{\rho\sqrt{2}}\left\{\frac{\sqrt{\rho - \cos\theta}}{\beta}\sin\theta + \sqrt{\rho + \cos\theta}\cos\theta\right\}. \tag{9}$$

Dependence (9) is presented in Fig. 4. The values $V = 0$, $0.7c$, $0.8c$, $0.9c$ correspond to curves 1-4. As for the stresses σ_{yz}, for $V < c/\sqrt{3}$ the stresses $\sigma_{\theta z}$ have a maximum in the angular distribution at $\theta = 0$. For $V > c/\sqrt{3}$ the maximum is observed for the values $\theta = \pm\theta_*$, where θ_* is determined as follows:

$$\cos^2\theta_* = \frac{5}{3} - \frac{c^2}{V^2} - \frac{2}{3}\sqrt{1 + 12\frac{c^2}{V^2}}\cos\left(\frac{\gamma + \pi}{3}\right), \tag{10}$$

where

$$\cos\gamma = -\frac{1 - 90c^2/V^2 + 54c^4/V^4}{(1 + 12c^2/V^2)^{3/2}}.$$

The dependence of θ_* on the parameter V/c determined by the relationship (10) is presented in Fig. 3 (curve 2). As the ratio V/c changes from $1/\sqrt{3}$ to $1/\sqrt{2}$, θ_* varies between 52°24' and 90°. In the domain $1/\sqrt{2} < V/c < 1$ the values of θ_* lie in the domain $\theta_* > 90°$, and we have $\theta_* = 90°$ for $V/c = 1$.

Let us calculate the rate of energy liberation G for an individual crack. It is known [9] that the rate of energy liberation for a longitudinal shear crack is related to the stress intensity factor by the relationship

$$G = \frac{\pi}{2\mu}\frac{K^2}{\sqrt{1 - V^2/c^2}}.$$

Substituting the value of the stress intensity factor from (8), we obtain $G = \tau^2_0 d/\mu$, i.e., in this problem the rate of energy liberation is explicitly independent of the rate of crack growth.

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